A CONTINUUM OF TOTALLY INCOMPARABLE HEREDITARILY INDECOMPOSABLE BANACH SPACES

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Abstract. A family is constructed of cardinality equal to the continuum, whose members are totally incomparable Hereditarily Indecomposable Banach spaces.

1. Introduction

All Banach spaces considered in this paper are real, infinite dimensional. By a *subspace* of a Banach space we shall mean an infinite dimensional, closed linear subspace. A Banach space is said to be *Hereditarily Indecomposable* (H.I.) if for every pair Y, Z of subspaces of X with $Y \cap Z = \{\mathbf{0}\}$, the subspace $Y + Z$ is not closed. The famous example of Gowers and Maurey [14] of a Banach space without unconditional basic sequence, was observed by W. Johnson to be H.I. Since the appearance of the Gowers-Maurey space the study of H.I. spaces has been one of the most important research topics in modern Banach space theory. We refer to [23] and [6] for a detailed survey of results.

It is proved in [6] that every Banach space not containing an isomorph of ℓ_1 has a subspace which is a quotient of an H.I. space. A recent result of S. Argyros [8] states that a separable Banach space universal for the class of reflexive H.I. spaces, is also universal for the class of separable Banach spaces. Both results indicate the large variety of H.I. spaces. The aim of this paper is towards this direction. Our main result is the following:

Theorem 1.1. *There exists a family of cardinality equal to the continuum whose members are totally incomparable, reflexive H.I. spaces.*

Recall that the Banach spaces X and Y are *totally incomparable* if no subspace of X is isomorphic to a subspace of Y.

The construction of H.I. spaces is not an easy task. The crucial step was Schlumprecht's construction of an arbitrarily distortable Banach space [27]. Recall that the Banach space $(X, \|\cdot\|)$ is *arbitrarily distortable* if for every $\lambda > 1$, there exists an equivalent norm $|\cdot|$ on X so that for every subspace Y of X there exist non-zero vectors x, y in Y such that $\|x\| = \|y\|$, yet $|x|/|y| > \lambda$. Schlumprecht's space had an immense impact

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in the development of the theory because of its connection to the Gowers-Maurey construction, as well as to the solution of the distortion problem for ℓ_p, $1 < p < \infty$, [24].

The first example of an arbitrarily distortable, asymptotic ℓ_1 space was given in [3]. They showed that there exist infinite subsets $M = (m_i)$, $N = (n_i)$ of $\mathbb{N}$ so that the mixed Tsirelson space $T(\frac{1}{m_i}, S_{n_i})_{i=1}^{\infty}$, is arbitrarily distortable. In the same paper this example was conditionalized to yield an asymptotic ℓ_1 H.I. space.

The proof of Theorem 1.1 is based on ideas from [3]. However, our argument is considerably simpler. We shall next describe how this paper is organized. In Section 3 we introduce, for a given scalar $d > 1$, the infinite subsets N and P of $\mathbb{N}$ and the null scalar sequence $\mathbf{a}$, the $(d, N, P, \mathbf{a})$ distortion property, Definition 3.1, of an asymptotic ℓ_1 Banach space. This property will enable us to give a criterion, Theorem 3.2, for an an asymptotic ℓ_1 Banach space to be arbitrarily distortable. We also show how to obtain totally incomparable arbitrarily distortable spaces. We apply Theorem 3.2 in Section 4 in order to give an alternative proof of the fact that certain mixed Tsirelson spaces are arbitrarily distortable [3], [2], [5]. These spaces can be described as the completion of c_{00}, the space of all ultimately vanishing real sequences, under the norm given by $\|x\| = \sup\{\sum_{i=1}^{\infty} \mu(\{i\})x(i) : \mu \in \mathcal{M}\}$, where $\mathcal{M}$ is a suitable symmetric subset of the finitely supported signed measures on $\mathbb{N}$ containing the point mass measures and closed under interval restrictions. The main difficulty in the study of mixed Tsirelson spaces is that the norming set $\mathcal{M}$ is defined by means of an inductive procedure. We are able to by pass this difficulty by describing $\mathcal{M}$ analytically and proving a decomposition result for its members, Lemma 4.3, which greatly simplifies the argument for the distortion of $T(\frac{1}{m_i}, S_{n_i})_{i=1}^{\infty}$.

In Section 5, we choose a subset $\mathcal{N}$ of $\mathcal{M}$ which is maximal with respect to a Maurey-Rosenthal type of condition [19] and show in Theorem 3.5 that the completion of c_{00} under the norm induced by $\mathcal{N}$ is an H.I. space satisfying a $(d, N, P, \mathbf{a})$ distortion property. Various choices of $\mathcal{N}$ give rise to totally incomparable H.I. spaces.

In order to prove that a space X is H.I., we employ Theorem 3.6 which loosely speaking asserts that if for every $\epsilon > 0$ there exist integers $k < n$ such that every block subspace Y of X contains a sufficiently large (in the Schreier sense) block basis $z_1 < \cdots < z_p$ with the property that $\|\sum_{i=1}^{p} a_i z_i\| \geq \epsilon \|\sum_{i=1}^{p} a_i e_i\|_n$, whenever $(a_i)_{i=1}^{p} \subset \mathbb{R}^+$, while $\|\sum_{i=1}^{p} a_i z_i\| \leq \|\sum_{i=1}^{p} a_i e_i\|_{Ck}$, for every sequence $(a_i)_{i=1}^{p}$ in $\mathbb{R}$, then X contains no infinite unconditional sequence. In the above, (e_i) is the natural unit vector basis of c_{00} and $\|\cdot\|_n$, $\|\cdot\|_{Ck}$ denote the n-th Schreier and k-th conditional Schreier norms respectively.

The precise statements for the results mentioned above are given in Section 3. The proof of Theorem 1.1, presented in Section 3, follows from Theorem 3.5 and Proposition 3.3 combined with two fundamental results of

descriptive set theory, the infinite Ramsey theorem [10], [22] and a theorem of Kuratowski [17].

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [18]. If D is any set, we let $[D]$ (resp. $[D]^{<\infty}$) denote the set of its infinite (resp. finite) subsets. Given $M \in [\mathbb{N}]$, the notation $M = (m_i)$ indicates that $M = \{m_1 < m_2 < \cdots\}$. Let E and F be finite subsets of $\mathbb{N}$. We write $E < F$ if $\max E < \min F$.

Suppose now that X is a Banach space with a Schauder basis (e_n). A sequence (u_n) in X is a *block basis* of (e_n) if there exist successive subsets $F_1 < F_2 < \cdots$ of $\mathbb{N}$ and a scalar sequence (a_n) so that $u_n = \sum_{i \in F_n} a_i e_i$, for every $n \in \mathbb{N}$. We adopt the notation $u_1 < u_2 < \cdots$ to indicate that (u_n) is a block basis of (e_n). We let $\mathrm{supp} u_n$ denote the set $\{i \in F_n : a_i \neq 0\}$. The *range* $r(u_n)$ of u_n, is the smallest integer interval containing $\mathrm{supp} u_n$. The subspace of X generated by a block basis of (e_n) is called a *block subspace*. We next review two important hierarchies. The Schreier hierarchy $\{S_\xi\}_{\xi<\omega_1}$, [1] and the repeated averages hierarchy, $(\xi_n^M)_{n=1}^\infty$, $\xi < \omega_1$, $M \in [\mathbb{N}]$, [4]. Since we shall only be using the families $\{S_\xi\}_{\xi<\omega}$, and $(\xi_n^M)_{n=1}^\infty$, $\xi < \omega$, $M \in [\mathbb{N}]$, we confine the definitions to the finite ordinal case.

The Schreier families. We let $S_0 = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}$. Suppose S_ξ has been defined, $\xi < \omega$. We set

$$S_{\xi+1} = \{\cup_{i=1}^n F_i : n \in \mathbb{N},\, n \leq \min F_1, F_1 < \cdots < F_n,\, F_i \in S_\xi\, (i \leq n)\} \cup \{\emptyset\}.$$

An important property shared by the Schreier families is that they are *hereditary*: If $F \in S_\xi$ and $G \subset F$, then $G \in S_\xi$. Another important property is that they are spreading: If $\{p_1, \cdots, p_k\} \in S_\xi$, $p_1 < \cdots < p_k$, and $q_1 < \cdots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \cdots, q_k\} \in S_\xi$. It is not hard to check that if $F_1 < \cdots < F_n$ are members of S_α such that $\{\min F_i : i \leq n\}$ belongs to S_β, then $\cup_{i=1}^n F_i$ belongs to $S_{\alpha+\beta}$.

The repeated averages hierarchy. We first let (e_n) denote the unit vector basis of c_{00}. Given $\xi < \omega$ and $M \in [\mathbb{N}]$, we define by induction, a sequence $(\xi_n^M)_{n=1}^\infty$ of finitely supported probability measures on $\mathbb{N}$ whose supports are successive subsets of M.

If $\xi = 0$, then $\xi_n^M = e_{m_n}$, for all $n \in \mathbb{N}$, where $M = (m_n)$.

Assume that $(\xi_n^M)_{n=1}^\infty$ has been defined for all $M \in [\mathbb{N}]$. Set

$$[\xi + 1]_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \xi_i^M$$

where $m_1 = \min M$. Suppose that $[\xi+1]_1^M < \cdots < [\xi+1]_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max \mathrm{supp}\, [\xi+1]_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$[\xi+1]^M_{n+1} = \frac{1}{k_n}\sum_{i=1}^{k_n} \xi_i^{M_n}.$$

It follows that $\operatorname{supp}\xi_n^M$ belongs to S_ξ, and moreover it is a maximal (under inclusion) member of S_ξ. It can be easily shown, by induction, that if i and j belong to $\operatorname{supp}\xi_n^M$ and $i<j$, then $\xi_n^M(\{i\}) \geq \xi_n^M(\{j\})$.

For a probability measure μ in $\mathbb{N}$ and $\xi<\omega$, we set $\|\mu\|_\xi = \sup\{\mu(F) : F \in S_\xi\}$. It is proven in [13], [7] that $\|\xi_1^M\|_{\xi-1} \leq \frac{\xi}{\min M}$, for every $\xi \geq 1$ and $M \in [\mathbb{N}]$. It follows that for every $P \in [\mathbb{N}]$, every $\xi \geq 1$ and every $\epsilon > 0$, there exists $M \in [P]$ such that $\|\xi_1^M\|_{\xi-1} < \epsilon$. This property of the repeated averages will be very useful in the sequel. For a detailed study of these hierarchies we refer to [1], [4], [25], [12], [7] and [13].

We continue by introducing some more terminology. A finite collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is said to be rS_ξ-admissible, $\xi<\omega$, $r\in\mathbb{N}$, if there exists an enumeration $\{I_k : k \leq n\}$ of $\mathcal{F}$ such that $I_1 < \cdots < I_n$ and the set $\{\min I_k : k \leq n\}$ is the union of r members of S_ξ. In case $\{\min I_k : k\leq n\}$ is a maximal (under inclusion) member of S_ξ, $\mathcal{F}$ is called maximally S_ξ-admissible. A finite block basis $u_1 < \cdots < u_n$ in a Banach space with a basis is rS_ξ (resp. maximally S_ξ)-admissible, if $\{\operatorname{supp} u_i : i \leq n\}$ is.

In what follows, X is a Banach space with a basis (e_n). The support of every block basis of (e_n) will always be taken with respect to (e_n).

Definition 2.1. *Let (u_n) be a normalized block basis of (e_n), $\epsilon>0$ and $1 \leq \xi < \omega$. Set $p_n = \min\operatorname{supp} u_n$, $n \in \mathbb{N}$, and $P=(p_n)$.*

1. *A generic (ϵ,ξ) average of (u_n) is any vector of the form $\sum_{n=1}^\infty \xi_1^R(p_n)u_n$, where $R\in[P]$ and $\|\xi_1^R\|_{\xi-1}<\epsilon$.*
2. *An (ϵ,ξ) average of (u_n) is any generic (ϵ,ξ) average of a normalized block basis of (u_n).*
3. *A normalized (ϵ,ξ) average of (u_n) is any vector u of the form $u=\frac{v}{\|v\|}$, where v is an (ϵ,ξ) average of (u_n). In case $\|v\| \geq \frac{1}{2}$, u is a smoothly normalized (ϵ,ξ) average of (u_n).*

Notation . *Let E^* be a finite collection of successive intervals of $\mathbb{N}$ and let u be a finite linear combination of (e_n).*

1. *We let $I(u,E^*)$ denote the number of elements of E^* which are intersected by $\operatorname{supp} u$.*
2. *Let D be a finite block basis of (e_n) such that the support of every member of D intersects at least one member of E^*. We set $D(E^*,1) = \{u \in D : I(u,E^*)=1\}$ and $D(E^*,2)=\{u\in D : I(u,E^*)\geq 2\}$.*

Before closing this section, we recall the definitions of the Schreier space, X^ξ, and conditional Schreier space, CX^ξ, $\xi<\omega$. X^ξ is the completion of c_{00} under the norm $\|x\|_\xi = \sup\{\sum_{i\in F}|x(i)| : F\in S_\xi\}$. X^0 is isometric to c_0. X^1 was introduced by Schreier [28] in order to provide an example of a

weakly null sequence without Cesaro summable subsequence. The generalized family of Schreier spaces $\{X^\xi\}_{\xi<\omega_1}$ was studied in [1], where it is shown that the natural Schauder basis (e_n) of X^ξ is 1-unconditional and shrinking. For a detailed study of the spaces $\{X^\xi\}_{\xi<\omega}$ we refer to [13].

The conditional Schreier spaces $\{CX^\xi\}_{\xi<\omega}$, were constructed by H. Rosenthal (unpublished). CX^ξ is the completion of c_{00} under the norm

$$\|x\|_{C\xi} = \sup\left\{\sum_{k=1}^{n}\left|\sum_{i\in J_k} x(i)\right| : n\in\mathbb{N},\ (J_k)_{k=1}^n \text{ are } S_\xi \text{ admissible intervals }\right\}.$$

The natural basis (e_n) of CX^ξ is of course, a conditional basis. When $\xi=0$, (e_n) is equivalent to the summing basis of c_0. We also mention the following useful fact: Suppose $(a_i)_{i=1}^n$ is a non-increasing finite sequence of non-negative scalars. Then $\|\sum_{i=1}^n(-1)^i a_i e_{t_i}\|_{C\xi} \le \|\sum_{i=1}^n a_i e_{t_i}\|_\xi$, for every increasing sequence of integers $(t_i)_{i=1}^n$.

3. Main results

We start this section by recalling that a normalized sequence (x_n) in a Banach space is an ϵ-ℓ_1^ξ *spreading model*, $\epsilon>0$, if $\|\sum_{i\in F} a_i x_i\| \ge \epsilon\sum_{i\in F}|a_i|$, for every $F\in S_\xi$ and all choices of scalars $(a_i)_{i\in F}$.

A Banach space X with a basis (e_n) is *asymptotic* ϵ-ℓ_1^ξ, $1\le\xi<\omega$, if every normalized block basis of (e_n) is an ϵ-ℓ_1^ξ spreading model. X is asymptotic ℓ_1, if it is asymptotic ϵ-ℓ_1^1, for some $\epsilon>0$ [20]. For an asymptotic ϵ-ℓ_1^ξ space X with a basis (e_n) and $\delta>0$, we define

$$\begin{aligned}\tau(X,\delta)=\sup\{\zeta<\omega :\ & \text{every normalized block basis of } (e_n)\\ & \text{has a subsequence which is a } \delta-\ell_1^\zeta \text{ spreading model }\}.\end{aligned}$$

Evidently, $\tau(X,\epsilon)\ge\xi$. The modulus $\tau(X,\delta)$ is implicitly defined in [25] and [2]. Of course $\tau(X,\delta)$ depends on the choice of the basis (e_n), but it will be clear from the context which basis is used. In case U is a block subspace of X, $\tau(U,\delta)$ will be calculated with respect to the block basis that generates U.

Definition 3.1. *Let X be a Banach space with a basis (e_i). Let $N=(n_i)$ and $P=(p_i)$ be infinite subsets of $\mathbb{N}$ such that $n_{i-1}\le p_i<\frac{n_i}{2}$, for every $i\in\mathbb{N}$. Let $\mathbf{a}=(\delta_i)$ be a decreasing null sequence of scalars, and let $d>1$. X is said to satisfy the $(d,N,P,\mathbf{a})$ distortion property if for every $j\in\mathbb{N}$, X is an asymptotic δ_j-$\ell_1^{n_j}$ space such that $\tau(U,d\delta_j)<p_j$, for every block subspace U of X.*

Theorem 3.2. *Let $(X,\|\cdot\|)$ be a Banach space with a normalized, shrinking, bimonotone basis (e_i). Suppose that there exist N, P in $[\mathbb{N}]$, a scalar sequence $\mathbf{a}=(\delta_i)$ and $d>1$ so that X satisfies the $(d,N,P,\mathbf{a})$ distortion property. Then X is arbitrarily distortable.*

Proof. In the sequel the admissibility of every block basis of (e_i) will always be considered with respect to (e_i). Given $j \in \mathbb{N}$, we set

$$\mathcal{A}_j = \Big\{ \delta_j \sum_{i=1}^{k} x_i^* : (x_i^*)_{i=1}^k \subset B_{X^*} \text{ is } S_{n_j} - \text{admissible} \Big\}.$$

In the above, the admissibility of (x_i^*) is measured with respect to (e_i^*), the sequence of functionals biorthogonal to (e_i). Because $\tau(X, \delta_j) \geq n_j$, we have that $\mathcal{A}_j \subset B_{X^*}$. Indeed, suppose that $\delta_j \sum_{i=1}^k x_i^* \in \mathcal{A}_j$ and let $x \in X$, $\|x\| \leq 1$. Put $x_i = x|\mathrm{ran}(x_i^*)$, $i \leq k$. Since (e_i) is bimonotone, $\| \sum_{i=1}^k x_i \| \leq 1$. Furthermore, $(x_i)_{i=1}^k$ is S_{n_j} admissible. Hence, $\delta_j \sum_{i=1}^k \|x_i\| \leq 1$ and the assertion follows.

We define an equivalent norm $\| \cdot \|_j$ on X in the following manner:

$$\|x\|_j = \delta_j \|x\| + \sup\{x^*(x) : x^* \in \mathcal{A}_j\}.$$

Let (u_i) be a normalized block basis of (e_i), and let $j_0 \in \mathbb{N}$. Let U be the block subspace of X generated by (u_i). Since $\tau(U, d\delta_{j_0}) < p_{j_0}$, there exists a normalized block basis (v_i) of (e_i) in U having no subsequence which is a $d\delta_{j_0}$-$\ell_1^{p_{j_0}}$ spreading model. It follows, by the main result of [12] combined with Corollary 3.6 of [7], that there exists a subsequence $(v_i)_{i \in M}$ of (v_i) such that for every $x^* \in B_{X^*}$, the block basis $V_{x^*} = \{v_i : i \in M, \, |x^*(v_i)| \geq 8d\delta_{j_0}\}$, is $S_{p_{j_0}}$ admissible.

We next choose v_0, a generic (δ_{j_0}, n_{j_0}) average of $(v_i)_{i \in M}$. It is easily seen that for some $x_0^* \in \mathcal{A}_{j_0}$ we have that $x_0^*(v_0) \geq \delta_{j_0}$. Therefore, $\|v_0\|_{j_0} \geq \delta_{j_0}$. On the other hand, V_{x^*} is $S_{p_{j_0}}$ admissible, for every $x^* \in B_{X^*}$ and $p_{j_0} < n_{j_0}$. It follows that $\|v_0\| \leq (8d+1)\delta_{j_0}$. We let $v = \frac{v_0}{\|v_0\|}$ and observe that $\|v\|_{j_0} \geq \frac{1}{8d+1}$.

Let now $j > j_0$. Arguing similarly, we can find a normalized block basis (w_i) of (u_i) and a generic $({\delta_j}^2, n_j)$ average w_0 of (w_i) such that $v < w_0$ and $\delta_j \leq \|w_0\| \leq (8d+1)\delta_j$. We let $w = \frac{w_0}{\|w_0\|}$. We are going to show that $\|w\|_{j_0} \leq (8d+5)\delta_{j_0}$. Suppose that $\delta_{j_0} \sum_{i=1}^k x_i^* \in \mathcal{A}_{j_0}$, and let E^* denote the collection of the ranges of the x_i^*'s. Let $D = \{w_r : |\sum_{i=1}^k x_i^*(w_r)| \geq 8d\delta_j\}$. Observe that by the choice of (w_i) we have that $D(E^*, 1)$ is $2S_{n_{j_0}+p_j}$ admissible. On the other hand $D(E^*, 2)$ is $2S_{n_{j_0}}$ admissible and thus D is $4S_{2p_j}$ admissible. Because $2p_j < n_j$, we obtain the estimate $\sum_{i=1}^k x_i^*(w_0) \leq (8d+4)\delta_j$. Hence, $\|w\|_{j_0} \leq (8d+5)\delta_{j_0}$, as claimed. Finally, $\frac{\|v\|_{j_0}}{\|w\|_{j_0}} \geq \frac{1}{(8d+1)(8d+5)\delta_{j_0}}$. The proof is now complete since j_0 was arbitrary. $\square$

Proposition 3.3. *Let X_r have a shrinking basis $(e_k^r)_{k=1}^\infty$, $r = 1, 2$. Assume that X_r satisfies the $(d_r, N_r, P_r, \mathbf{a})$ distortion property, $r = 1, 2$, and that $\mathbf{a} = (\delta_i)$ satisfies $\lim_i \frac{\delta_{i+1}}{\delta_i} = 0$. Suppose that for every $i_0 \in \mathbb{N}$ there exist $i > j > i_0$ such that $n_i^1 = n_j^2$, where $N_r = (n_k^r)_{k=1}^\infty$, $r = 1, 2$. Then X_1 and X_2 are totally incomparable.*

Proof. Suppose the assertion is false. A standard perturbation argument yields a normalized block basis (u_k) of (e^1_k) equivalent to a block basis (w_k) of (e^2_k). Let T be an isomorphism from $[(u_k)]$ onto $[(w_k)]$ such that $T(u_k) = w_k$, for all $k \in \mathbb{N}$. We can choose $i_0 \in \mathbb{N}$ such that $\frac{\delta_{i+1}}{\delta_i} < \frac{1}{d_1\|T\|\|T^{-1}\|}$, for every $i \geq i_0$. Our assumptions allow us to choose $i > j > i_0$ such that $n^1_i = n^2_j$. Let (v_k) be a normalized block basis of (u_k) having no subsequence which is a $d_1\delta_i$-$\ell_1^{n^1_i}$ spreading model. But since $(T(v_k))$ is a block basis of (w_k), it follows that for every $F \in S_{n^2_j}$ and all choices of scalars $(a_k)_{k\in F}$

$$\Big\|\sum_{k\in F} a_k T(v_k)\Big\| \geq \frac{\delta_j}{\|T^{-1}\|}\sum_{k\in F}|a_k|.$$

Hence, $\|\sum_{k\in F} a_k v_k\| \geq \frac{\delta_j}{\|T\|\|T^{-1}\|}\sum_{k\in F}|a_k|$, for every $F \in S_{n^2_j}$ and all choices of scalars $(a_k)_{k\in F}$. However, $\frac{\delta_i}{\delta_j} \leq \frac{\delta_{j+1}}{\delta_j}$, and therefore $\frac{\delta_j}{\|T\|\|T^{-1}\|} > d_1\delta_i$. Thus, (v_k) is a $d_1\delta_i$-$\ell_1^{n^1_i}$ spreading model contrary to our assumptions. □

Definition 3.4. *Let $M = (m_i) \in [\mathbb{N}]$ such that $m_1 > 6$ and $m_i^2 < m_{i+1}$, for all $i \in \mathbb{N}$. Choose $L \in [\mathbb{N}]$, $L = (l_i)$ such that $l_1 > 4$ and $2^{l_i} > m_i$, for all $i \in \mathbb{N}$. The infinite subset $N = (n_i)$ of $\mathbb{N}$ is said to be M-good, if $l_j(f^N_j + 1) < n_j$, for all $j \in \mathbb{N}$. In the above, (f^N_j) is the sequence given by $f^N_1 = 1$ while for $j \geq 2$,*

$$f^N_j = \max\Big\{\sum_{i<j}\rho_i n_i : \rho_i \in \mathbb{N}\cup\{0\}\ (i<j),\ \prod_{i<j} m_i^{\rho_i} < m_j^3\Big\}.$$

Note that f^N_j is well defined because $m_1 > 1$. It is easy to see that for every $P \in [\mathbb{N}]$ there exists $N \in [P]$ which is M-good. The main result of Section 5 is the following

Theorem 3.5. *Suppose $N = (n_i)$ is M-good. Set $N^{(2)} = (n_{2i})$, $F^{(2)} = (f^N_{2i} + 2)$ and $\mathbf{a} = (\frac{1}{m_{2i}})$. Then there exists a reflexive H.I. space $X(N)$ satisfying the $(6, N^{(2)}, F^{(2)}, \mathbf{a})$ distortion property.*

The proof is given in Section 5. We now pass to the

Proof of Theorem 1.1. We first choose $N_0 \in [\mathbb{N}]$ such that every $N \in [N_0]$ is M-good. To see that such a N_0 exists, set

$$\mathcal{D} = \{N \in [\mathbb{N}] : N \text{ is } M-\text{good }\}.$$

We can easily verify that $\mathcal{D}$ is closed in the topology of pointwise convergence in $[\mathbb{N}]$, and therefore it is a Ramsey set. Because $\mathcal{D}\cap[R] \neq \emptyset$, for every $R \in [\mathbb{N}]$, the infinite Ramsey theorem yields $N_0 \in [\mathbb{N}]$ such that $[N_0] \subset \mathcal{D}$, as claimed.

It is a well known fact that $[N_0]$ endowed with the topology of pointwise convergence is a perfect Polish space. We let $[N_0]^2 = [N_0] \times [N_0]$ and set

$$G = \{(N, R) \in [N_0]^2,\ N = (n_i),\ R = (r_i) | \forall i_0 \in \mathbb{N},\ \exists i > j > i_0 : n_{2i} = r_{2j}\}.$$

A straightforward application of the Baire category theorem yields that G is a dense G_δ subset of $[N_0] \times [N_0]$. By a result of Kuratowski [17] and Mycielski [21] (cf. [16], p. 129, Theorem 19.1, or Proposition 3.6 of [13]), there exists $C \subset [N_0]$ homeomorphic to the Cantor set such that $(N_1, N_2) \in G$, whenever N_1, N_2 are distinct elements of C.

We can now apply Theorem 3.5 to obtain a family $\{X(N) : N \in C\}$ of reflexive H.I. spaces such that for every $N \in C$, $X(N)$ satisfies the $(6, N^{(2)}, F^{(2)}, \mathbf{a})$ distortion property, where $N^{(2)}$, $F^{(2)}$ and $\mathbf{a}$ are as in the statement of Theorem 3.5. Since $(N_1, N_2) \in G$ whenever N_1 and N_2 are distinct elements of C, Proposition 3.3 implies that $X(N_1)$ and $X(N_2)$ are totally incomparable. The proof of the theorem is now complete. $\square$

To construct H.I. spaces we shall make use of the following

Theorem 3.6. *Let X be a Banach space with a basis (x_i). Let (n_j), (k_j) be increasing sequences of positive integers such that $k_j < n_j$, for all $j \in \mathbb{N}$, and let (δ_j) be a null sequence of positive scalars. Assume that for every block subspace Y of X and every $j \in \mathbb{N}$ there exists a block basis $z_1 < \cdots < z_p$ of (x_i) in Y such that letting $t_i = \min \operatorname{supp} z_i$, $i \le p$, the following are satisfied:*

1. *$\{t_i : i \le p\}$ is a maximal S_{n_j} set and $\| \sum_{i=1}^p a_i z_i \| \ge c_1 \delta_j \| \sum_{i=1}^p a_i e_{t_i} \|_{n_j}$, for every sequence $(a_i)_{i=1}^p$ in $\mathbb{R}^+$.*
2. *$\| \sum_{i=1}^p a_i z_i \| \le c_2 \| \sum_{i=1}^p a_i e_{t_i} \|_{C k_j} + c_3 {\delta_j}^2$, for every sequence $(a_i)_{i=1}^p$ in $\mathbb{R}$ with $\sum_{i=1}^p |a_i| \le 1$,*

where c_1, c_2 and c_3 are absolute positive constants. Then X has no infinite unconditional sequence. If moreover, given Y, Z block subspaces of X and $j \in \mathbb{N}$, such a block basis $(z_i)_{i=1}^p$ can be found with the additional property that $z_i \in Y$, if i is odd, while $z_i \in Z$, if i is even, then X is H.I.

Proof. Let (u_i) be an infinite block basis of (x_i), and let $j \in \mathbb{N}$. Set $P = \{p_i : i \in \mathbb{N}\}$, where $p_i = \min \operatorname{supp} u_i$. We can find $R \in [P]$ such that $\| [n_j]_1^L \|_{k_j} < {\delta_j}^2$, for every $L \in [R]$. Let $Y = [u_i : p_i \in R]$. Choose $z_1 < \cdots < z_p$ in Y, according to the hypothesis. There exists $L \in [R]$ such that $\{t_i : i \le p\} = \operatorname{supp} [n_j]_1^L$. Put $a_i = [n_j]_1^L(t_i)$, $i \le p$, and note that $(a_i)_{i=1}^p$ is non-increasing. We now have that

$$\left\| \sum_{i=1}^p a_i z_i \right\| \ge c_1 \delta_j \left\| \sum_{i=1}^p a_i e_{t_i} \right\|_{n_j} = c_1 \delta_j.$$

On the other hand,

$$\left\| \sum_{i=1}^p (-1)^i a_i z_i \right\| \le c_2 \left\| \sum_{i=1}^p a_i e_{t_i} \right\|_{k_j} + c_3 {\delta_j}^2,$$

as $(a_i)_{i=1}^p$ is non-increasing. Hence,

$$\Big\|\sum_{i=1}^p (-1)^i a_i z_i\Big\| \le (c_2+c_3)\delta_j{}^2 \le \frac{c_2+c_3}{c_1}\delta_j \|\sum_{i=1}^p a_i z_i\|.$$

Since j was arbitrary, (u_i) is not unconditional. The moreover statement is immediate. □

4. Mixed Tsirelson spaces

Recall that if $\mathcal{M}$ is a set of finitely supported signed measures on $\mathbb{N}$ which satisfies the following:

1. $e_n^* \in \mathcal{M}$, for all $n \in \mathbb{N}$, where e_n^* denotes the point mass measure at n.
2. $\mathcal{M}$ is symmetric i.e., if $\mu \in \mathcal{M}$ then $-\mu \in \mathcal{M}$,
3. $\mathcal{M}$ is pointwise bounded, that is $\mu(\{n\}) \le 1$, for every $\mu \in \mathcal{M}$,
4. $\mathcal{M}$ is closed under restriction to initial segments i.e., if $\mu \in \mathcal{M}$, then $\mu|\{1,\dots,n\} \in \mathcal{M}$,

then one can define a norm $\|\cdot\|_{\mathcal{M}}$ on c_{00} in the following manner:

$$\|\sum_{i=1}^\infty a_i e_i\|_{\mathcal{M}} = \sup\{\sum_{i=1}^\infty a_i \mu(\{i\}) : \mu \in \mathcal{M}\},$$

for every finitely supported scalar sequence (a_i). Of course, (e_i) is the natural basis of c_{00}. Letting $X_{\mathcal{M}}$ denote the completion of $(c_{00}, \|\cdot\|_{\mathcal{M}})$, we see that (e_n) is a normalized, monotone basis for $X_{\mathcal{M}}$. In case $\mu|J \in \mathcal{M}$, for every $\mu \in \mathcal{M}$ and $J \subset \mathbb{N}$, then (e_n) is 1-unconditional and bimonotone.

The main result of this section is

Theorem 4.1. *Suppose N is M-good. There exists $\mathcal{M}$, a set of finitely supported signed measures on $\mathbb{N}$ satisfying conditions 1-4, above, and such that the following properties are fulfilled:*

1. *(e_n) is an 1-unconditional, shrinking, bimonotone basis for $X_{\mathcal{M}}$.*
2. *$X_{\mathcal{M}}$ satisfies the $(6, N, P, \mathbf{a})$ distortion property, where $P = (f_i^N + 2)$ and $\mathbf{a} = (\frac{1}{m_i})$.*

We first give the construction of $\mathcal{M}$ and prove a number of lemmas necessary for the proof of Theorem 4.1.

Construction of $\mathcal{M}$. Given $M = (m_i)$, $N = (n_i)$, with N being M-good, we construct $\mathcal{M}$, a set of signed measures on $\mathbb{N}$ in the following manner: Let

$$\mathcal{D} = \big\{(t_1,\dots,t_{3n}) : n \in \mathbb{N}, t_{3i-2} \in M\ (i<n),\ t_{3n-2}=0,$$
$$t_{3i-1} \in [\mathbb{N}]^{<\infty} \setminus \{\emptyset\},\ t_{3i} \in \{-1,1\}\ (i \le n)\big\}.$$

Given $F \in \mathcal{D}^{<\infty}$, $F \ne \emptyset$, we let $\mathcal{T}_F$ denote the set of all tuples of length divisible by 3 which are initial segments of elements of F. We can partially order the elements of $\mathcal{T}_F$ by initial segment inclusion and thus $\mathcal{T}_F$ becomes a finite tree with terminal nodes precisely the members of F. Given $\alpha \in \mathcal{T}_F$ then $m \in M$ is an *M-entry* of α, if $m \in \alpha$. We shall denote the last three entries of α by m_α, I_α and ϵ_α respectively. A rooted tree $\mathcal{T} = \mathcal{T}_F$ (a tree

is rooted if it has a unique root), is said to be *appropriate* provided the following properties hold:

1. If $\alpha \in \mathcal{T}$ is terminal, then $I_\alpha = \{p_\alpha\}$, for some $p_\alpha \in \mathbb{N}$.
2. If $\alpha \in \mathcal{T}$ is non-terminal and $m_\alpha = m_j$, for some $j \in \mathbb{N}$, then $(I_\beta)_{\beta \in D_\alpha}$ is S_{n_j}-admissible and $I_\alpha = \cup_{\beta \in D_\alpha} I_\beta$. Here D_α stands for the set of the immediate successors of α in $\mathcal{T}$.

We set

$$\mathcal{G} = \{\mathcal{T} : \mathcal{T} \text{ is an appropriate tree } \}.$$

We make the convention that the empty tree belongs to $\mathcal{G}$.

Notation . *Let $\mathcal{T} \in \mathcal{G}$ and $\alpha \in \mathcal{T}$.*

1. *α^- stands for the predecessor of α in $\mathcal{T}$. In case α is the root of $\mathcal{T}$ we put $\alpha^- = \emptyset$.*
2. *$|\alpha|$ is the length of α. Thus, $|\alpha| = 3n$ if $\alpha = (t_1, \dots, t_{3n})$. We now define $o(\mathcal{T}) = \max\{|\beta| : \beta \in \mathcal{T}\}$, the height of the tree $\mathcal{T}$.*
3. *$m(\alpha) = \prod_{m_i \in \alpha^-} m_i$. We set $m(\alpha) = 1$ if $|\alpha| = 3$.*
4. *$n(\alpha) = \sum_{m_i \in \alpha^-} n_i$. We set $n(\alpha) = 0$, if $|\alpha| = 3$.*

Given $\mathcal{T} \in \mathcal{G}$, set

$$\mu_{\mathcal{T}} = \sum_{\alpha \in max\mathcal{T}} m(\alpha)^{-1} \epsilon(\alpha) \epsilon_\alpha e^*_{p_\alpha},$$

where $max\mathcal{T}$ is the set of terminal nodes of $\mathcal{T}$ and $I_\alpha = \{p_\alpha\}$ for $\alpha \in max\mathcal{T}$. We have also set $\epsilon(\alpha) = \prod_{\beta < \alpha} \epsilon_\alpha$ for $\alpha \in \mathcal{T}$. We make the convention $\epsilon(\alpha) = 1$, if $|\alpha| = 3$. We also set $\mu_\emptyset = 0$. Of course, $\mu_{\mathcal{T}}$ is a finitely supported signed measure on $\mathbb{N}$ whose support is equal to I_{α_0}, where α_0 is the root of $\mathcal{T}$. We also observe that $|\mu_{\mathcal{T}}(\{n\})| \leq 1$, for all $n \in \mathbb{N}$.

We finally set $\mathcal{M} = \{\mu_{\mathcal{T}} : \mathcal{T} \in \mathcal{G}\}$. Note that $e^*_n \in \mathcal{M}$ as $\big\{(0, \{n\}, 1)\big\} \in \mathcal{G}$. We shall introduce some more notation in order to investigate properties of the set $\mathcal{M}$.

Notation . *Let $\mathcal{T} \in \mathcal{G}$ and let α_0 denote its root.*

1. *Given $\alpha \in \mathcal{T}$ set $\mathcal{T}_\alpha = \{\beta \setminus \alpha^- : \beta \in \mathcal{T}, \alpha \leq \beta\}$. Clearly, $\mathcal{T}_\alpha \in \mathcal{G}$.*
2. *We let $w(\mathcal{T}) = 1$, if $|\mathcal{T}| = 1$. In case $m_{\alpha_0} \in M$, we set $w(\mathcal{T}) = m_{\alpha_0}$.*
3. *Let $J \subset \mathbb{N}$. We let $\mathcal{T}|J$ denote the tree resulting from $\mathcal{T}$ by keeping only those $\alpha \in \mathcal{T}$ for which $I_\alpha \cap J \neq \emptyset$ and replacing I_α by $I_\alpha \cap J$. It is easy to see that $\mathcal{T}|J \in \mathcal{G}$.*
4. *We let $-\mathcal{T}$ denote the tree resulting from $\mathcal{T}$ by changing ϵ_{α_0} to $-\epsilon_{\alpha_0}$. Clearly, $-\mathcal{T} \in \mathcal{G}$ and moreover $\mu_{-\mathcal{T}} = -\mu_{\mathcal{T}}$.*

Remark . *Let $\mathcal{T} \in \mathcal{G}$.*

1. *If $J \subset \mathbb{N}$, then $\mu_{\mathcal{T}|J} = \mu_{\mathcal{T}}|J$.*
2. *If $\alpha \in \mathcal{T}$ then $m(\alpha)\epsilon(\alpha)\mu_{\mathcal{T}}|I_\alpha = \mu_{\mathcal{T}_\alpha}$.*

Remark . *Suppose $\mathcal{T}_i \in \mathcal{G}$, $i \leq n$. Let α_i be the root of $\mathcal{T}_i$, $i \leq n$. We shall say that $\{\mathcal{T}_i : i \leq n\}$ is S_ξ-admissible, $\xi < \omega$, if $\{I_{\alpha_i} : i \leq n\}$ is. We shall also write $\mathcal{T}_1 < \cdots < \mathcal{T}_n$ if $I_{\alpha_1} < \cdots < I_{\alpha_n}$. It is easy to see that if $\mathcal{T}_1 < \cdots < \mathcal{T}_n$ is S_{n_j}-admissible then $\frac{\sum_{i=1}^n \mu_{\mathcal{T}_i}}{m_j} \in \mathcal{M}$.*

It follows by our preceding remarks that $\mathcal{M}$ is pointwise bounded, symmetric and closed under restriction to subsets of $\mathbb{N}$. Hence (e_n) is an 1-unconditional, bimonotone basis for $X_{\mathcal{M}}$. It is not hard to check that $X_{\mathcal{M}}$ is isometric to $T(\frac{1}{m_i}, S_{n_i})_{i=1}^{\infty}$. We also obtain by our preceding remarks that if $(x_i)_{i=1}^k$ is an S_{n_j}-admissible block basis of (e_n) then $\|\sum_{i=1}^k x_i\| \geq \frac{1}{m_j}\sum_{i=1}^k \|x_i\|$. Hence $X_{\mathcal{M}}$ is an asymptotic $\frac{1}{m_j}$-$\ell_1^{n_j}$ space. It follows that (e_n) is boundedly complete. Let now ν be a w^*-cluster point of $\mathcal{M}$. Using the reflexivity argument of [3] (cf. also [29]), one obtains that for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\||\nu|\,[e_i : i \geq k]\| < \epsilon$. It follows from this that (e_n) is shrinking and thus $X_{\mathcal{M}}$ is reflexive.

Remark . *Suppose (u_n) is a normalized block basis of (e_n) and u an (ϵ, n_j) average of (u_n). Then $\frac{1}{m_j} \leq \|u\| \leq 1$.*

Lemma 4.2. *Let $\mathcal{T} \in \mathcal{G}$. Let F be a subset of $\mathcal{T}$ consisting of pairwise incomparable nodes. Then $\{I_\alpha : \alpha \in F\}$ is S_p-admissible, where $p = \max\{n(\alpha) : \alpha \in F\}$.*

Proof. By induction on $o(\mathcal{T})$. If $o(\mathcal{T}) = 3$ the assertion of the lemma is trivial. Assuming the assertion true when $o(\mathcal{T}) < 3k$, $k > 1$, let $\mathcal{T} \in \mathcal{G}$ with $o(\mathcal{T}) = 3k$. If $|F| = 1$ there is nothing to prove. So assume $|F| \geq 2$. Let α_0 be the root of $\mathcal{T}$ and let $w(\mathcal{T}) = m_i$ for some $i \in \mathbb{N}$. We denote by D the set of immediate successors of α_0 in $\mathcal{T}$. Given $\alpha \in D$ let $F_\alpha = \{\beta \in F : \alpha \leq \beta\}$. Because $o(\mathcal{T}_\alpha) \leq 3k - 3$ we can apply the induction hypothesis on $\mathcal{T}_\alpha$ and the set $\{\beta \setminus \alpha^- : \beta \in F_\alpha\}$ to deduce that the collection $\{I_\beta : \beta \in F_\alpha\}$ is S_{p_1}-admissible, where $p_1 = \max\{n(\beta \setminus \alpha^-) : \beta \in F_\alpha\}$. Since $n(\beta \setminus \alpha^-) = n(\beta) - n(\alpha)$ and $n(\alpha) = n_i$ whenever $\alpha \in D$, we obtain that $\{I_\beta : \beta \in F_\alpha\}$ is S_{p-n_i}-admissible, for every $\alpha \in D$. But also, $\{I_\alpha : \alpha \in D\}$ is S_{n_i}-admissible whence $\{I_\alpha : \alpha \in F\}$ is S_p-admissible. $\square$

To simplify our notation, we set $f_j = f_j^N$. We make the following observation: Let $\mathcal{T} \in \mathcal{G}$ and let $\alpha \in \mathcal{T}$. Assume that $m(\alpha) < m_j^3$ and that all M-entries of α^- are smaller than m_j. Then $n(\alpha) \leq f_j$. Our next lemma will be crucial for the proof of the main result.

Lemma 4.3 (Decomposition Lemma)**.** *Let $\mathcal{T}_0 \in \mathcal{G}$. Let $j \in \mathbb{N}$ such that $w(\mathcal{T}_0) < m_j$. Then there exist an S_{f_j}-admissible subset $\mathcal{G}_0$ of $\mathcal{G}$ and a scalar sequence $(\lambda_{\mathcal{T}})_{\mathcal{T} \in \mathcal{G}_0}$ in $[-1, 1]$ so that the following are satisfied:*

1. $\mu_{\mathcal{T}_0} = \sum_{\mathcal{T} \in \mathcal{G}_0} \lambda_{\mathcal{T}} \mu_{\mathcal{T}}$.
2. *For each $\mathcal{T} \in \mathcal{G}_0$ at least one of the following hold: either $w(\mathcal{T}) = 1$ (thus $\mu_{\mathcal{T}} = \pm e^*_{\mathcal{T}(p)}$ for some $\mathcal{T}(p) \in \mathbb{N}$), or $w(\mathcal{T}) \geq m_j$, or $|\lambda_{\mathcal{T}}| \leq \frac{1}{m_j^2}$.*

Proof. Let $\mathfrak{B}$ denote the set of all branches of $\mathcal{T}_0$ (a branch is a maximal well ordered subset of $\mathcal{T}_0$). If $w(\mathcal{T}_0) = 1$ the assertion is trivial. So assume that $w(\mathcal{T}_0) = m_{i_0}$ for some $i_0 < j$. Given $b \in \mathfrak{B}$ set

$$\alpha^1(b) = \max\{\beta \in b : m(\beta) < m_j^2 \text{ and if } m_i \in \beta^- \text{ then } i < j\}.$$

Note that $\alpha^1(b)$ is well defined and that $(m_{i_0}, I, \epsilon) < \alpha^1(b)$ since $i_0 < j$ ((m_{i_0}, I, ϵ) being the root of $\mathcal{T}_0$).

Let us say that $b \in \mathfrak{B}$ is of type 1 if $\alpha^1(b)$ is terminal in $\mathcal{T}_0$. If b is not of type 1 then it is of type 2 (resp. 3), if the last M-entry of $\alpha^1(b)$ is greater than or equal (resp. smaller than) m_j. We then denote by $\alpha^2(b)$ the immediate successor of $\alpha^1(b)$ in b.

We let $A_1 = \{\alpha^1(b) : b \in \mathfrak{B} \text{ is of type } 1\}$, $A_2 = \{\alpha^1(b) : b \in \mathfrak{B} \text{ of type } 2\}$ and $A_3 = \{\alpha^2(b) : b \in \mathfrak{B} \text{ is of type } 3\}$. Observe that the following properties hold:

1. If $\alpha \in A_3$ then all M-entries of α^- are smaller than m_j, $m(\alpha^-) < m_j^2$, yet $m_j^2 \leq m(\alpha) < m_j^3$.
2. If $\alpha \in A_2$, then α is non-terminal, all M-entries in α^- are smaller than m_j, the last M-entry of α is greater than or equal to m_j and $m(\alpha) < m_j^2$.
3. If $\alpha \in A_1$ then α is terminal, all M-entries in α^- are smaller than m_j and $m(\alpha) < m_j^2$.

It is not hard to check now that $A = \cup_{t=1}^3 A_t$ consists of pairwise incomparable nodes of $\mathcal{T}_0$ and hence $\{I_\alpha : \alpha \in A\}$ consists of successive subsets of $\mathbb{N}$. Moreover, $I = \cup\{I_\alpha : \alpha \in A\}$. Because $m(\alpha) < m_j^3$ and all M-entries of α^- are smaller than m_j whenever $\alpha \in A$, we obtain that $n(\alpha) \leq f_j$ for all $\alpha \in A$. Lemma 4.2 now yields that $\{I_\alpha : \alpha \in A\}$ is S_{f_j}-admissible. Finally, we let $\mathcal{G}_0 = \{(\mathcal{T}_0)_\alpha : \alpha \in A\}$. Since $m(\alpha)\epsilon(\alpha)\mu_{\mathcal{T}_0}|I_\alpha = \mu_{(\mathcal{T}_0)_\alpha}$, for all $\alpha \in \mathcal{T}$, we set $\lambda_{(\mathcal{T}_0)_\alpha} = \frac{1}{m(\alpha)\epsilon(\alpha)}$ for $\alpha \in A$. We can easily verify that the desired properties hold. $\square$

In the sequel, we shall be using a variety of block bases of (e_n). The support of each of them will always be taken with respect to (e_n).

Lemma 4.4. *Let (u_n) be a normalized block basis of (e_n). Let $j \in \mathbb{N}$, $j \geq 2$ and let u be a generic $(\epsilon, f_j + 1)$ average of (u_n) with $\epsilon < \frac{1}{2m_j}$. Let $i < j$ and let $\mathcal{T}_1 < \cdots < \mathcal{T}_t$ in $\mathcal{G}$ be S_{n_i}-admissible. Then $\sum_{k=1}^t \mu_{\mathcal{T}_k}(u) \leq 2$. In particular, $\mu_{\mathcal{T}}(u) \leq \frac{2}{w(\mathcal{T})}$, if $w(\mathcal{T}) < m_j$.*

Proof. Observe that $\frac{1}{m_j}\sum_{k=1}^t \mu_{\mathcal{T}_k} \in \mathcal{M}$ and hence $\sum_{k=1}^t \mu_{\mathcal{T}_k}(u_n) \leq m_j$, for all $n \in \mathbb{N}$. Let $P = (p_n)$, where $p_n = \min \operatorname{supp} u_n$. Set $\xi = f_j + 1$ and suppose that $u = \sum_{n=1}^\infty \xi_1^R(p_n)u_n$, for some $R \in [P]$. Let E^* denote the collection of the ranges of the $\mu_{\mathcal{T}_k}$'s, and let D denote the collection of those u_n's whose support intersects at least one member of E^*. Put $I_r = \{n \in \mathbb{N} : u_n \in D(E^*, r)\}$, $r = 1, 2$. Because $D(E^*, 2)$ is $2S_{n_i}$-admissible

and $n_i \leq f_j$, we obtain that $\sum_{k=1}^t \mu_{\mathcal{T}_k}(\sum_{n\in I_2} \xi_1^R(p_n)u_n) \leq m_j 2\epsilon$. On the other hand we clearly have that $\sum_{k=1}^t \mu_{\mathcal{T}_k}(\sum_{n\in I_1} \xi_1^R(p_n)u_n) \leq 1$. Thus, $\sum_{k=1}^t \mu_{\mathcal{T}_k}(u) \leq 2$. □

Lemma 4.5. *Let (u_n) be a normalized block basis of (e_n). Let $\epsilon > 0$ and $j \in \mathbb{N}$. Then there exists a smoothly normalized $(\epsilon, f_j + 1)$ average of (u_n).*

Proof. Let $P = (p_n)$, where $p_n = \min\operatorname{supp} u_n$ for $n \in \mathbb{N}$. We can assume without loss of generality that $\|\xi_1^R\|_{\xi-1} < \epsilon$ for every $R \in [P]$ where $\xi = f_j + 1$. We are going to show that there exists a normalized block basis of (u_n) admitting a generic (ϵ, ξ) average of norm at least $\frac{1}{2}$. Suppose instead that this were false. Then it is easy to construct for every $1 \leq r \leq l_j$, a block basis (u_i^r) of (u_i) so that letting $p_i^r = \min\operatorname{supp} u_i^r$ and $P_r = (p_i^r)$ the following are satisfied:

1. (u_i^r) is a block basis of (u_i^{r-1}). $(u_i^0 = u_i)$
2. $u_i^r = \sum_{n=1}^\infty \xi_i^{P_{r-1}}(p_n^{r-1})\frac{u_n^{r-1}}{\|u_n^{r-1}\|}$, for all $i \in \mathbb{N}$. $(p_n^0 = p_n)$
3. $\|u_i^r\| < \frac{1}{2}$, for all $i \in \mathbb{N}$.
4. For every $i \in \mathbb{N}$, if $u_i^r = \sum_{n\in F_i^r} a_n u_n$ with $a_n > 0$ for $n \in F_i^r$, then $\sum_{n\in F_i^r} a_n \geq 2^{r-1}$ and $(u_n)_{n\in F_i^r}$ is $S_{\xi r}$-admissible.

The construction is easily done by induction. Taking $r = l_j$ we see from 3. that $\|u_i^{l_j}\| < \frac{1}{2}$. On the other hand 4. implies that $\|u_i^{l_j}\| \geq \frac{2^{l_j-1}}{m_j}$ as $\xi l_j < n_j$. Thus, $m_j > 2^{l_j}$ contradicting the choice of l_j. □

Our next lemma yields that $X_{\mathcal{M}}$ satisfies the $(6, N, F, \mathbf{a})$ distortion property where $F = (f_i + 2)$ and $\mathbf{a} = (\frac{1}{m_i})$.

Lemma 4.6. *Let (u_j) be a normalized block basis of (e_j). Suppose that (y_j) is a block basis of (u_j) so that y_j is a smoothly normalized (ϵ_j, f_j+1) average of (u_j) with $\epsilon_j < \frac{1}{2m_j}$. Given $j_0 \in \mathbb{N}$ and $J_0 \in [\mathbb{N}]$, there exists $J \in [J_0]$ such that $j_0 < \min J$ and for every $\mathcal{T} \in \mathcal{G}$, $D_{\mathcal{T}} = \{y_j : j \in J, |\mu_{\mathcal{T}}(y_j)| \geq \frac{5}{m_{j_0}}\}$ is $S_{f_{j_0}+1}$-admissible.*

Proof. Note first that Lemma 4.5 guarantees the existence of the block basis (y_j). Let $P = (p_j)_{j\in J_0}$, where $p_j = \min\operatorname{supp} y_j$. By passing to a subsequence of $(y_j)_{j\in J_0}$, if necessary, we can assume that the union of any 4 $S_{f_{j_0}}$ subsets of P belongs to $S_{f_{j_0}+1}$. Choose $J \in [J_0]$, $J = (j_i)$, such that $j_0 < j_1$ and $\|y_{j_i}\|_{\ell_1} < \frac{m_{j_{i+1}}}{m_{j_i}}$, for every $i \geq 2$ (if $v = \sum_{i=1}^n a_i e_i$, then $\|v\|_{\ell_1} = \sum_{i=1}^n |a_i|$).

Let $\mathcal{T}_0 \in \mathcal{G}$. Suppose first that $w(\mathcal{T}_0) \geq m_{j_0}$. We show that in this case $|D_{\mathcal{T}_0}| \leq 1$. Indeed, suppose first that $w(\mathcal{T}_0) < m_{j_1}$. Lemma 4.4 yields that $|\mu_{\mathcal{T}_0}(y_j)| \leq \frac{4}{m_{j_0}}$ for all $j \in J$ whence $D_{\mathcal{T}_0} = \emptyset$.

If $w(\mathcal{T}_0) \geq m_{j_1}$ choose $s \geq 2$ so that $m_{j_{s-1}} \leq w(\mathcal{T}_0) < m_{j_s}$. Observe that if $1 \leq i < s-1$ then

$$|\mu_{\mathcal{T}_0}(y_{j_i})| \leq \frac{1}{w(\mathcal{T}_0)}\|y_{j_i}\|_{\ell_1} < \frac{1}{w(\mathcal{T}_0)}\frac{m_{j_{s-1}}}{m_{j_i}} < \frac{1}{m_{j_0}}.$$

When $i \geq s$, Lemma 4.4 yields $|\mu_{\mathcal{T}_0}(y_{j_i})| \leq \frac{4}{w(\mathcal{T}_0)} < \frac{4}{m_{j_0}}$. Hence $D_{\mathcal{T}_0} \subset \{y_{j_{s-1}}\}$ and so our claim holds.

The final case to consider is that of $w(\mathcal{T}_0) < m_{j_0}$. Clearly, $D_{\mathcal{T}_0} = \emptyset$, if $w(\mathcal{T}_0) = 1$. We employ the decomposition Lemma 4.3 to find an $S_{f_{j_0}}$ admissible subset $\mathcal{G}_0$ of $\mathcal{G}$ and scalars $(\lambda_{\mathcal{T}})_{\mathcal{T}\in\mathcal{G}_0}$ satisfying the conclusion of Lemma 4.3. Let E^* denote the collection of the ranges of the $\mu_{\mathcal{T}}$'s ($\mathcal{T} \in \mathcal{G}_0$). Our previous work implies that $D_{\mathcal{T}_0}(E^*, 1)$ is $2S_{f_{j_0}}$ admissible. But also, $D_{\mathcal{T}_0}(E^*, 2)$ is $2S_{f_{j_0}}$ admissible since $\mathcal{G}_0$ is $S_{f_{j_0}}$ admissible. It follows that $D_{\mathcal{T}_0}$ is $S_{f_{j_0}+1}$ admissible. □

Proof of Theorem 4.1. Let U be a block subspace of $X_{\mathcal{M}}$ spanned by the normalized block basis (u_j) of (e_j). Let $j_0 \in \mathbb{N}$ and choose a block basis $(y_j)_{j\in J}$ of (u_j) satisfying the conclusion of Lemma 4.6. Applying Corollary 3.4 of [2] (cf. also Corollary 3.3 of [12]), we obtain that for every subsequence of $(y_j)_{j\in J}$ which is a δ-$\ell_1^{f_{j_0}+2}$ spreading model, it must be the case that $\delta \leq \frac{5}{m_{j_0}}$ and thus $\tau(U, \frac{6}{m_{j_0}}) < f_{j_0} + 2$. □

Terminology. Let j_0 and $(y_j)_{j\in J}$ satisfy the conclusion of Lemma 4.6. Every normalized (ϵ, n_{j_0}) average u of $(u_j)_{j=1}^{\infty}$ of the form $u = \frac{v}{\|v\|}$, where v is a generic (ϵ, n_{j_0}) average of $(y_j)_{j\in J}$, will be called a *normalized* (ϵ, n_{j_0}) *average of* $(u_j)_{j=1}^{\infty}$ *resulting from Lemma 4.6*. Note that Lemmas 4.5 and 4.6 guarantee the existence of such averages for every block basis $(u_j)_{j=1}^{\infty}$.

Corollary 4.7. *Let (y_j) satisfy the assumptions of Lemma 4.6. Given $j_0 \in \mathbb{N}$ and $J_0 \in [\mathbb{N}]$, there exists $J \in [J_0]$ such that $j_0 < \min J$ and for every $\mathcal{T}_0 \in \mathcal{G}$, $w(\mathcal{T}_0) \neq m_{j_0}$, $D_{\mathcal{T}_0} = \{y_j : j \in J, |\mu_{\mathcal{T}}(y_j)| \geq \frac{5}{m_{j_0} m_e}\}$ is $S_{f_{j_0}+1}$-admissible, where we have set $m_e = \min\{m_{j_0}, w(\mathcal{T}_0)\}$.*

Proof. We choose J_0 and j_0 as we did in the proof of Lemma 4.6. Suppose first that $w(\mathcal{T}_0) > m_{j_0}$. Because $m_i^2 < m_{i+1}$, the argument in the proof of Lemma 4.6 shows that $|D_{\mathcal{T}_0}| \leq 1$.

When $w(\mathcal{T}_0) < m_{j_0}$, we apply the decomposition Lemma 4.3 to find an $S_{f_{j_0}}$ admissible subset $\mathcal{G}_0$ of $\mathcal{G}$ and scalars $(\lambda_{\mathcal{T}})_{\mathcal{T}\in\mathcal{G}_0}$ satisfying the conclusion of Lemma 4.3. Note that if $\mathcal{T} \in \mathcal{G}_0$ and $w(\mathcal{T}) = m_{j_0}$, then $|\lambda_{\mathcal{T}}| \leq \frac{1}{w(\mathcal{T}_0)}$ and thus for all $j \in J$, $|\lambda_{\mathcal{T}}\mu_{\mathcal{T}}(y_j)| \leq \frac{4}{m_{j_0} m_e}$, by Lemma 4.4. Using the splitting argument of Lemma 4.6, we conclude that $D_{\mathcal{T}_0}$ is $S_{f_{j_0}+1}$-admissible. □

Corollary 4.8. *Let u be a normalized (ϵ, n_{j_0}) average of $(u_j)_{j=1}^{\infty}$ resulting from Lemma 4.6 with $\epsilon \leq \frac{1}{12 m_{j_0}^2}$. Let $\mathcal{G}_0$ be an S_{n_i}-admissible subset of $\mathcal{G}$, $i < j_0$, such that $m_{j_0} \notin \{w(\mathcal{T}) : \mathcal{T} \in \mathcal{G}_0\}$. Then, $|\sum_{\mathcal{T}\in\mathcal{G}_0} \mu_{\mathcal{T}}(u)| \leq \frac{6}{m_e}$, where $m_e = \min\{w(\mathcal{T}) : \mathcal{T} \in \mathcal{G}_0\} \cup \{m_{j_0}\}$.*

Proof. Set $\xi = n_{j_0}$. Let $p_j = \min \operatorname{supp} y_j$, $j \in J$ and $P = \{p_j : j \in J\}$. There exists $R \in [P]$ so that $u = \frac{v}{\|v\|}$, where $v = \sum_{j\in J} \xi_1^R(p_j) y_j$ and

$\|\xi_1^R\|_{n_{j_0}-1} < \epsilon$. Note that $\|v\| \geq \frac{1}{m_{j_0}}$. Applying a splitting argument similar to that of Lemma 4.6 and taking in account Corollary 4.7, we obtain that $\{y_j : j \in J, |\sum_{\mathcal{T}\in\mathcal{G}_0} \mu_{\mathcal{T}}(y_j)| \geq \frac{5}{m_{j_0}m_e}\}$ is $3S_{2f_{j_0}+1}$-admissible. The assertion follows from Lemma 4.4 and the fact that $2f_{j_0}+1 < n_{j_0}$. $\square$

Remark . *It is easy to see that in case $w(\mathcal{T}) = 1$, for all $\mathcal{T} \in \mathcal{G}_0$, one obtains the estimate $|\sum_{\mathcal{T}\in\mathcal{G}_0} \mu_{\mathcal{T}}(u)| \leq \frac{1}{m_{j_0}}$.*

5. Hereditarily indecomposable spaces

This section is devoted to the proof of Theorem 3.5. Recall that X is H.I. if and only if, for every pair of subspaces Y, Z of X and every $\epsilon > 0$, there exist $y \in Y$ and $z \in Z$ so that $y \neq z$ and $\|y - z\| \leq \epsilon\|y + z\|$.

Let $M \in [\mathbb{N}]$, $M = (m_i)$ and let $N \in [\mathbb{N}]$, $N = (n_i)$, which is M-good. Let $\mathcal{M}$ be the set of measures constructed in the previous section by using the sets M and N. We shall choose $\mathcal{N} \subset \mathcal{M}$ so that the resulting space $X_{\mathcal{N}}$ is a reflexive H.I. space satisfying the conclusion of Theorem 3.5.

We can find an injection

$$\sigma\colon \{\mathcal{T}_1 < \cdots < \mathcal{T}_n : n \in \mathbb{N}, \mathcal{T}_i \in \mathcal{G}\,(i \leq n)\} \to \{m_{2j} : j \in \mathbb{N}\}$$

so that $\sigma(\mathcal{T}_1, \ldots, \mathcal{T}_n) > w(\mathcal{T}_i)$, for all $i \leq n$.

Definition 5.1. 1. *An S_p-admissible sequence $\mathcal{T}_1 < \cdots < \mathcal{T}_n$ in $\mathcal{G}$ is said to be S_p-dependent, $p \in \mathbb{N}$, if $w(\mathcal{T}_1) = m_{2j_1}$, for some $j_1 > \frac{p}{2}$, and $\sigma(\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}) = w(\mathcal{T}_i)$, for all $2 \leq i \leq n$.*
2. *Let $\mathcal{T}_1 < \cdots < \mathcal{T}_n$ in $\mathcal{G}$, $p \in \mathbb{N}$ and $\mathcal{G}_0 \subset \mathcal{G}$. We shall say that $\mathcal{T}_1 < \cdots < \mathcal{T}_n$ admits an S_p-dependent extension in $\mathcal{G}_0$, if there exist $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and an S_p-dependent sequence $\mathcal{R}_1 < \cdots < \mathcal{R}_{n+k}$ in $\mathcal{G}_0$ so that $\mathcal{R}_{k+i}|[l,\infty) = \mathcal{T}_i$, for all $i \leq n$.*
3. *A subset $\mathcal{G}_0$ of $\mathcal{G}$ is said to be self-dependent, if the following property is satisfied for every $\mathcal{T} \in \mathcal{G}_0$: Let $\alpha \in \mathcal{T}$ so that its last M-entry equals m_{2j+1} for some $j \in \mathbb{N}$. Let D_α denote the set of immediate successors of α in $\mathcal{T}$. Then $\{\mathcal{T}_\beta : \beta \in D_\alpha\}$ admits an $S_{n_{2j+1}}$-dependent extension in $\mathcal{G}_0$.*

Definition 5.2. *We let $\mathfrak{D}$ denote the union of all non-empty, self-dependent, symmetric and closed under restriction to intervals, subsets of $\mathcal{G}$. Recall that $\mathcal{G}_0 \subset \mathcal{G}$ is symmetric if $-\mathcal{T} \in \mathcal{G}_0$ whenever $\mathcal{T} \in \mathcal{G}_0$. $\mathcal{G}_0$ is closed under interval restrictions if $\mathcal{T}|J \in \mathcal{G}_0$ whenever $\mathcal{T} \in \mathcal{G}_0$ and J is an interval.*

Of course $\mathfrak{D}$ is a maximal, under inclusion, subset of $\mathcal{G}$ with respect to the aforementioned properties. Set $\mathcal{N} = \{\mu_{\mathcal{T}} : \mathcal{T} \in \mathfrak{D}\}$. We will show that $X_{\mathcal{N}}$ is H.I.

Remark . *The maximality of $\mathfrak{D}$ implies the following:*

1. *$e_n^* \in \mathcal{N}$, for all $n \in \mathbb{N}$.*

2. *If $\mathcal{T} \in \mathfrak{D}$, then $\mathcal{T}_\alpha \in \mathfrak{D}$, for all $\alpha \in \mathcal{T}$ and so the decomposition Lemma 4.3 holds for $\mathfrak{D}$.*
3. *If $\mathcal{T}_1 < \cdots < \mathcal{T}_k$ in $\mathfrak{D}$ is $S_{n_{2i}}$-admissible, $i \in \mathbb{N}$, then $\frac{\mu_{\mathcal{T}_1} + \cdots + \mu_{\mathcal{T}_k}}{m_{2i}} \in \mathcal{N}$.*
4. *If $\mathcal{T}_1 < \cdots < \mathcal{T}_k$ in $\mathfrak{D}$ is $S_{n_{2i+1}}$-dependent, $i \in \mathbb{N}$, then $\frac{\mu_{\mathcal{T}_1} + \cdots + \mu_{\mathcal{T}_k}}{m_{2i+1}} \in \mathcal{N}$.*
5. *Because of 3., all the results obtained in the previous section about (ϵ, ξ) averages in $X_{\mathcal{M}}$, where ξ is either n_j or $f_j + 1$ for some $j \in \mathbb{N}$, still hold in $X_{\mathcal{N}}$ provided j is even.*

Note that $X_{\mathcal{N}}$ is reflexive by the same argument that showed $X_{\mathcal{M}}$ is reflexive. Thus (e_i) is a shrinking basis for $X_{\mathcal{N}}$.

Proof of Theorem 3.5. It follows from Theorem 4.1 and our preceding remarks that $X_{\mathcal{N}}$ satisfies the $(6, N^{(2)}, F^{(2)}, \mathbf{a})$ distortion property. We show that $X_{\mathcal{N}}$ is H.I. This is accomplished through Theorem 3.6. Let (u_n) be a normalized block basis of (e_n) and let $j \in \mathbb{N}$. Set $P = (p_n)$, where $p_n = \min \operatorname{supp} u_n$. We can assume that the union of any 7 $S_{f_{2j+1}}$ subsets of P belongs to $S_{f_{2j+1}+1}$. Successive applications of Corollary 4.8 yield a normalized block basis $g_1 < \cdots < g_p$ of (u_n), $\mathcal{T}_1 < \cdots < \mathcal{T}_p$ in $\mathfrak{D}$, and integers $j_1 < \cdots < j_p$ with $2j+1 < j_1$, satisfying the following:

1. g_i is a normalized $(\frac{1}{12 m_{2j_i}^2}, n_{2j_i})$ average of (u_n) resulting from Lemma 4.6.
2. $w(\mathcal{T}_i) = m_{2j_i}$, $\operatorname{supp} \mu_{\mathcal{T}_i} \subset r(g_i)$ and $\mu_{\mathcal{T}_i}(g_i) > \frac{1}{2}$, for all $i \leq p$.
3. $\sigma(\mathcal{T}_1, \dots, \mathcal{T}_{i-1}) = w(\mathcal{T}_i)$, for all $i \leq p$.
4. $\{g_i : i \leq p\}$ is maximally $S_{n_{2j+1}}$-admissible.

Put $\theta_i = (\mu_{\mathcal{T}_i}(g_i))^{-1}$, $z_i = \theta_i g_i$, and note that $1 \leq \theta_i < 2$, $i \leq p$. We'll show that $(z_i)_{i=1}^p$ satisfies conditions 1. and 2. of Theorem 3.6, with $\delta_j = \frac{1}{m_{2j+1}}$, "$n_j$"$= n_{2j+1}$ and $k_j = f_{2j+1} + 1$. Condition 1. is immediate since $\mathcal{T}_1 < \cdots < \mathcal{T}_p$ is $S_{n_{2j+1}}$-dependent. Condition 2. is achieved by establishing the following

Claim: Given $\mathcal{T} \in \mathfrak{D}$, there exist intervals $J_1 < \cdots < J_s$ in $\{1, \dots, p\}$ so that

1. $\{z_{\min J_t} : t \leq s\}$ is $S_{f_{2j+1}+1}$-admissible.
2. $\mu_{\mathcal{T}} \mid \{z_i : i \in J_t\}$ is constant for all $t \leq s$.
3. $|\mu_{\mathcal{T}}(z_i)| < \frac{12}{m_{2j+1}^2}$, for all $i \notin \cup_{t=1}^s J_t$.

To prove the claim suppose first that $w(\mathcal{T}) > m_{2j+1}$. Corollary 4.8 yields that $|\mu_{\mathcal{T}}(z_i)| \geq \frac{12}{m_{2j+1}^2}$, for at most one $i \leq p$, and thus the claim holds in this case.

Next assume that $w(\mathcal{T}) = m_{2j+1}$. Without loss of generality, there exist an $S_{n_{2j+1}}$-dependent sequence $\mathcal{R}_1 < \cdots < \mathcal{R}_l$ in $\mathfrak{D}$ and an interval J so that $\mu_{\mathcal{T}} = \frac{1}{m_{2j+1}} \sum_{k=1}^l \mu_{\mathcal{R}_k} \mid J$. Let i_0 be the largest i for which $w(\mathcal{T}_i)$ is an element of $\{w(\mathcal{R}_k) : k \leq l\}$, or let $i_0 = 0$, if no such i exists. The injectivity of σ and Corollary 4.8 imply that if $i_0 \in \{0, 1\}$, or if $w(\mathcal{T}_{i_0}) = w(\mathcal{R}_1)$, then $|\mu_{\mathcal{T}}(z_i)| < \frac{12}{m_{2j+1}^2}$, for all $i \neq i_0$.

If $i_0 > 1$, then the injectivity of σ yields $w(\mathcal{T}_{i_0}) = w(\mathcal{R}_{i_0})$, $\mathcal{T}_i = \mathcal{R}_i$ for $i < i_0$ yet $\mathcal{T}_{i_0} \neq \mathcal{R}_{i_0}$. It follows now by Corollary 4.8, that $|\mu_{\mathcal{T}}(z_i)| < \frac{12}{m^2_{2j+1}}$, for all $i > i_0$. We also observe that there exists $i_1 < i_0$ such that $\mu_{\mathcal{T}}(z_i) = 0$, if $i < i_1$, while $\mu_{\mathcal{T}}(z_i) = \frac{1}{m_{2j+1}}$ if $i_1 < i < i_0 - 1$. Concluding, there exist four intervals $J_1 < J_2 < J_3 < J_4$ in $\{1, \dots, p\}$, some of which may possibly be empty, such that $\mu_{\mathcal{T}} \,|\, \{z_i : i \in J_t\}$ is constant for every $t \leq 4$, while $|\mu_{\mathcal{T}}(z_i)| < \frac{12}{m^2_{2j+1}}$, for each $i \notin \cup_{t=1}^4 J_t$.

Finally, assume $w(\mathcal{T}) < m_{2j+1}$. If $w(\mathcal{T}) = 1$, the claim trivially holds so suppose that $w(\mathcal{T}) > 1$. Choose $\mathcal{G}_0 \subset \mathfrak{D}$ $S_{f_{2j+1}}$-admissible and scalars $(\lambda_{\mathcal{R}})_{\mathcal{R} \in \mathcal{G}_0}$ according to the decomposition Lemma 4.3. By splitting the z_i's into two sets, those whose support intersects at least two of the ranges of the $\mu_{\mathcal{R}}$'s, and those whose support intersects at most one, we deduce from our previous work that there exist intervals $J_1 < \cdots < J_s$ in $\{1, \dots, p\}$ so that $\{z_{\min J_t} : t \leq s\}$ is $7S_{f_{2j+1}}$-admissible, $\mu_{\mathcal{T}} \,|\, \{z_i : i \in J_t\}$ is constant for all $t \leq s$, and $|\mu_{\mathcal{T}}(z_i)| < \frac{12}{m^2_{2j+1}}$, for all $i \notin \cup_{t=1}^s J_t$. Thus the claim holds and the proof is complete. $\square$

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